A Stylized Historical Introduction to the Classical Theory of the Electron

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By 1905, electron theory was widely regarded as the most promising extension of Maxwellian electrodynamics. A streamlined exposition of its basic hypotheses was provided by Max Abraham (1904) in a short article submitted in the spring of 1904. The paper articulates the theory on the basis of a small set of explicit physical hypotheses that jointly determine its kinematical and dynamical framework. In the spring of the following year, Abraham (1905) completed a more extensive presentation of electron theory in the second volume of the revised edition of August Föppl (1904) well-known treatise on electricity. Whereas the article offers a logically well-structured qualitative exposition, the book develops the theory in considerable mathematical detail, often in a manner that renders the logical relations among the different hypotheses less transparent. The aim of the following text is therefore to bring together these two presentations in a compact, semi-historical account that remains faithful to the original conceptual structure without being anachronistic, while presenting the material in a form accessible to contemporary readers who may be unfamiliar with Abraham's notation.

1 Lorentz–Maxwell Electrodynamics

A: <u>In space empty of matter and electricity, the Maxwell–Hertz equations hold</u>.

In their original interpretation, Maxwell equations were taken to single out a reference frame in which plane electromagnetic waves propagate isotropically with the same velocity $c = 3 \cdot 10^{10}\, cm/sec$. Motions referred to this privileged 'ether' frame were traditionally designated as *absolute motions*. In this frame, Maxwell's equations in the presence of charge density ρ and current density $\rho\mathbf{v}$ read

$$\text{rot } \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\rho\,\mathbf{v}, \qquad \text{div } \mathbf{B} = 0,$$

$$\text{rot } \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{div } \mathbf{E} = 4\pi\rho.$$

These equations determine the temporal evolution of the electric and magnetic field strengths **E** and **B** once the charge density ρ and its velocity field **v** are specified (subject to the continuity equation). Intuitively, they express that electric charges and currents, together with time-varying electric fields, generate magnetic fields, while time-varying magnetic fields generate electric fields. Electric fields begin and end on charges, magnetic fields have no sources or sinks, and the two fields are dynamically coupled so that a change in one induces the other.

These equations possess a crucial formal property: they are invariant under Lorentz transformations:

$$\begin{aligned} x' &= \gamma\,(x - vt), \\ t' &= \gamma\left(t - \frac{v}{c^2}\,x\right), \\ y' &= y, \\ z' &= z, \end{aligned} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}.$$

That is, if the fields **E** and **B** satisfy Maxwell's equations in the ether frame, then the appropriately transformed fields $\mathbf{E}'$ and $\mathbf{B}'$ by a Lorentz transformation satisfy equations of the same form in any other frame moving uniformly with respect to the ether frame.

Consider a Lorentz boost with velocity **v** along the x–axis. The transformed field components are

$$\begin{aligned} E'_x &= E_x, & B'_x &= B_x, \\ E'_y &= \gamma(E_y - vB_z)\,, & B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right), \\ E'_z &= \gamma(E_z + vB_y)\,, & B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{aligned}$$

Replacing unprimed by primed quantities, the equations retain their form. This invariance implies that the velocity c has the same value in all inertial frames, since Maxwell's equations contain the fixed constant c that determines the propagation speed of electromagnetic disturbances.

The mechanical properties of the electromagnetic field can be expressed algebraically in terms of **E** and **B**. The localized energy of Newtonian particles with definite positions in space is replaced by the energy *density* of the electromagnetic field, that is, the energy per unit volume:

$$W_{\text{EM}} = \frac{1}{2}\frac{1}{4\pi}\left(E^2 + B^2\right) = \frac{1}{8\pi}\left(E^2 + B^2\right).$$

The Poynting vector **S**, representing the energy flux density (the energy crossing a unit surface per unit time),[1] is given by

$$\mathbf{S} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{B}.$$

The cross product $\mathbf{E}\times\mathbf{B}$ vanishes if either field is absent or if they are parallel, and it is maximal when the fields are perpendicular, as in a plane electromagnetic wave. At each point, **S** points in the direction in which electromagnetic energy is transported, and its magnitude gives the amount of energy crossing a unit area per unit time.

[1] I use 'flux' in the sense of 'flux density' unless the context explicitly indicates a surface integral.

The electromagnetic field does not merely carry energy; it also exerts forces that can push, pull, or shear material bodies. The Maxwell stresses describe how the electromagnetic field exerts mechanical stresses—pressures and tensions—on matter and on itself. The components σ_{ij} $(i, j = 1, 2, 3)$ of the Maxwell stress tensor are given by, for example,

$$\begin{array}{ll} \text{Normal stress (longitudinal)} & \sigma_{zz} = \dfrac{1}{4\pi}\left(E_z^2 + B_z^2 - \tfrac{1}{2}(E^2 + B^2)\right) \\ \text{Normal stress (transverse)} & \sigma_{xx} = \dfrac{1}{4\pi}\left(E_x^2 + B_x^2 - \tfrac{1}{2}(E^2 + B^2)\right) \\ \text{Shear stress} & \sigma_{xz} = \dfrac{1}{4\pi}\left(E_x E_z + B_x B_z\right) \end{array}$$

Matter experiences electromagnetic forces wherever the field stresses are nonuniform. The divergence of the Maxwell stress tensor quantifies this nonuniformity in direct analogy with stress gradients in ordinary continua:

$$(\nabla \cdot \boldsymbol{\sigma})_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{1}$$

If the stress is spatially uniform, its divergence vanishes, and the internal forces on any local volume element are balanced. If the stress is nonuniform, its divergence represents a force density acting on matter (or, in vacuum, on the field itself), leading to acceleration or further deformation until equilibrium is established.

B: <u>Electricity consists of discrete positive and negative particles, called 'electron'</u>.

In classical electron theory, electrons are not conceived as pointlike entities, but as small yet finite regions of the ether in which the charge density ρ is nonvanishing. In many models, a mass density μ is likewise attributed to the electron, either as an intrinsic mechanical mass or as arising from the energy of its electromagnetic field. Since like charges repel, additional non-electromagnetic forces—often described as a material 'frame' or as cohesive stresses—must be postulated to stabilize the charge distribution and prevent it from dispersing under its own electrostatic repulsion.

Electrons are the carriers of charge within matter and thus account for the interaction between material bodies and the electromagnetic field in the ether. For uniform motion, the electromagnetic field associated with an electron forms a steady configuration that translates rigidly with it; no radiation is emitted. For accelerated motion, however, this steady state is disrupted. A portion of the Poynting flux then separates from the bound field and propagates outward as radiation, carrying energy to infinity. In this case, radiation-reaction effects arise, and the emitted fields travel through empty space, that is, through the pure ether.

Within this framework, electrical conduction in matter is explained by the drift of (typically negative) electrons through material bodies, thereby generating electric currents and electromagnetic fields in the ether. Polarization arises when bound electrons are displaced relative to oppositely charged constituents, producing electric fields. Magnetization is attributed to electrons circulating in closed orbits, which generate magnetic fields. Finally, oscillating electrons can absorb and re-emit electromagnetic energy, thereby accounting for the propagation of light through matter.

C: <u>Every electric current is, in this sense, a convective current of moving charged particles (in Lorentz's classical electron theory: electrons).</u>.

Classical electron theory assumes that all electrical currents—conduction, polarization, and magnetization—ultimately originate in the motion of electrons, and that no additional, irreducible current carriers need be postulated. The density of a convection current is given by the product $\rho\mathbf{v}$. Such a convection current generates the same magnetic field as the corresponding conduction current in Maxwell–Hertz theory. From hypotheses A, B, C the field equations follow, governing the time evolution of the electromagnetic field. Given a distribution of charges ρ and currents $\rho\mathbf{v}$ together with appropriate initial and boundary conditions for the fields, Maxwell's equations determine the subsequent values of $\mathbf{E}$ and $\mathbf{B}$ at every spatial point of the reference ether frame for times $t > 0$. A statement is supplemented to them concerning the force (at a given field) acting upon a volume element containing charge:

D: The electromagnetic force is additively composed of an electric part, independent of the charge's motion, and a magnetic part that depends on the charge's motion..

The *electromagnetic force density* $\mathbf{f}$ is the sum of an electric contribution, which acts on charges in an electric field, and a magnetic contribution, which acts on charges in motion:

$$\mathbf{f} = \rho\Big(\mathbf{E} + \mathbf{v} \times \mathbf{B}\Big)$$

The corresponding *total force* acting on a distribution of charges contained in a spatial region V is obtained by integration:

$$\mathbf{F} = \int_V \mathbf{f}\, dV$$

If the external fields $\mathbf{E}$ and $\mathbf{B}$ are approximately uniform over the volume V, and the charge moves with a common velocity $\mathbf{v}$, the integral reduces to:

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where $e = \int_V \rho\, dV$ is the total charge of the system.

If the positions and velocities of all electrons are known at a given instant t, hypotheses A–C determine the charge and current densities at that same instant. Specifying the electromagnetic field on an initial time-slice (or imposing a retarded/no-incoming-radiation condition) then fixes $\mathbf{E}$ and $\mathbf{B}$. Hypothesis D, together with Newton's fundamental law of motion, yields the accelerations of the electrons and thus completes the dynamical description.

Hypotheses A–D provide a closed set of equations for determining the coupled evolution of charged matter and the electromagnetic field. A corollary of these equations is the corresponding local conservation laws. When the field acts on an electron and accelerates it, energy is transferred from the electromagnetic field to the particle. Maxwell's equations then imply the local energy-balance equation (Poynting's theorem):

$$f_i v_i = -\frac{\partial w_{\text{EM}}}{\partial t} - \frac{\partial S_i}{\partial x_i}$$

Here $f_i v_i$ is the power transferred from the field to matter per unit volume (where f_i is the force density). The term $-\frac{\partial w_{\text{EM}}}{\partial t}$ represents the local decrease of the electromagnetic energy density w_{EM} stored in the field. The term $-\frac{\partial S_i}{\partial x_i} = -\nabla \cdot \mathbf{S}$ accounts for energy transport: it is positive when electromagnetic energy flows into the volume element from its surroundings and negative when energy flows out. For an accelerating charge that emits radiation, the energy carried away to large distances is described by the outward Poynting flux through a surrounding surface.

A mechanical force density also transfers *mechanical* momentum to the electron. When this force is integrated over an extended region containing charges and currents, the resulting change of mechanical momentum cannot, in general, be balanced by mechanical reactions within matter alone. Unless further structure is introduced, the interaction between matter and the electromagnetic field therefore appears to violate momentum conservation. This difficulty led Abraham to introduce an *electromagnetic momentum* associated with the field itself. Maxwell's equations imply that the electromagnetic field carries a momentum *density*

$$\mathbf{g}_{\mathrm{EM}} = \frac{\mathbf{S}}{c^2} = \frac{1}{c^2}\frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}$$

and that momentum is transported through space by electromagnetic stresses. In terms of momentum *flux*, selected components of the Maxwell stress tensor can be written as:

$$\begin{aligned}
&\text{Longitudinal momentum current} \quad && \sigma_{zz} = \frac{1}{4\pi}\left(E_z^2 + B_z^2 - \tfrac{1}{2}(E^2+B^2)\right)\\
&\text{Transverse momentum current} && \sigma_{xx} = \frac{1}{4\pi}\left(E_x^2 + B_x^2 - \tfrac{1}{2}(E^2+B^2)\right)\\
&\text{Shear momentum current} && \sigma_{xz} = \frac{1}{4\pi}\left(E_xE_z + B_xB_z\right)
\end{aligned}$$

where σ_{ij} denotes the flux of the i-component of electromagnetic momentum across a surface orthogonal to the j-direction.

Once these quantities are taken into account, the Lorentz force admits a reinterpretation as a local momentum–balance equation. The force acting on matter arises from the temporal change of electromagnetic momentum stored in the field and from the divergence of electromagnetic momentum fluxes. In component form, the Lorentz force density can be written as:

$$f_i = -\frac{\partial g_{\mathrm{el},i}}{\partial t} - \sum_j \frac{\partial \sigma_{ij}}{\partial x_j}$$

The left-hand side, f_i, represents the force density acting on the charges; if f_i is positive, the electrons are gaining momentum (i.e., accelerating). This gain is balanced by the two terms on the right-hand side. The first term represents the rate of decrease of the momentum density $g_{\mathrm{el},i}$ stored within the electromagnetic field itself. The second term—the negative divergence of the Maxwell stress tensor σ_{ij}—indicates the net rate at which momentum is transported into the local volume from the surrounding space. In this way, the Lorentz force law expresses the local exchange of momentum between charged matter and the electromagnetic field. Any gain or loss of mechanical momentum by matter is balanced by a corresponding change in electromagnetic momentum, which is either stored locally or transported through space via Maxwell stresses.

2 Abraham's Rigid Electron

The phenomena observed in cathode rays can be interpreted on the basis of hypothesis D if the electromagnetic force exerted by the external field is treated as an external force and if the free negative electrons constituting the cathode rays are ascribed an inertial mass m. At the same time, however, the fundamental hypotheses A–D entail that this mass cannot be regarded as purely mechanical. Since the electron generates an electromagnetic field, and since the field possesses energy and momentum, part — and in Abraham's program ultimately the whole — of the electron's inertia must be attributed to its own field.

An electron, by virtue of its charge, produces an electric field **E**, and by virtue of its motion a magnetic field **B**. When the electron is accelerated, these fields do not adjust instantaneously without dynamical consequence. Rather, the changing field configuration entails a redistribution of electromagnetic momentum which reacts back upon the charge itself. The resulting self-force opposes the acceleration, so that the resistance to changes of motion increases with velocity. In this respect the situation is analogous to hydrodynamics, where a body moving through a fluid entrains part of the surrounding medium and behaves as if it possessed an added mass. By analogy, the electromagnetic field surrounding the electron contributes to its effective inertia.

This line of reasoning suggested that the inertia of the electron might be determined entirely by its field energy and field momentum. The experiments of Kaufmann on the deflection of Becquerel (β-) rays in electric and magnetic fields indicated that, at sufficiently high velocities, the electromagnetic contribution to the electron's inertia becomes dominant. His measurements pointed to a pronounced dependence of the electron's mass on its velocity.

E: The electromagnetic forces of the external field and of the field generated by the electron itself are in equilibrium on the electron, in the sense of the mechanics of rigid bodies..

In rigid-body mechanics, equilibrium does not require the absence of forces, but the vanishing of their resultant and of the resultant torque. Applied to the electron, hypothesis E asserts that, at every instant, the total electromagnetic force and torque obtained by integrating the Lorentz force density over the volume of the electron balance internally, so that

$$F_{\text{EXT}} + F_{\text{SELF}} = 0,$$

and likewise for the corresponding torques. Under this condition, and together with the hypothesis of undeformability (F), the electron behaves dynamically as a mechanically rigid body, i.e. that their resultant force and torque vanish when taken over the electron as a whole.

F: The electron is not at all capable of any change of its shape..

Hypothesis F adds a kinematical constraint: the electron is *undeformable.* In Abraham's terminology (following Hertz), F is to be understood as a conditional equation: it does not require one to specify cohesive forces between volume elements, but implies only that whatever maintains rigidity performs no work and therefore need not enter the dynamical balance. On the basis of hypotheses A–F, the dynamics of an electron of arbitrary (but fixed) shape can be developed in a purely electromagnetic way, in the sense that the equations governing its motion as a whole are obtained from electromagnetic field energy–momentum relations, without the introduction of an independent mechanical law of motion. In particular, it is unnecessary to treat explicitly the forces holding together the volume elements of the electron, since by hypothesis these internal constraint forces perform no net work.

A resting electron possesses only electrostatic energy:

$$W_{\mathrm{EM}}^{0} = \frac{1}{8\pi} \int \mathbf{E}^2 \, dV.$$

When the electron moves with velocity v, a magnetic field $\mathbf{B}$ is generated and the electric field configuration is modified. The total electromagnetic field energy becomes

$$W_{\mathrm{EM}}(v) = \frac{1}{8\pi} \int \left(\mathbf{E}^2 + \mathbf{B}^2 \right) dV,$$

which exceeds the energy at rest. In uniform translation (within the quasi-stationary regime assumed in Abraham's electron dynamics) the field pattern translates steadily and no net energy is carried away to infinity as radiation. An external force accelerating the electron must therefore supply the increase of electromagnetic field energy associated with the changed state of motion, and progressively more work is required to produce the same change in speed as v increases. Dynamically, the electron behaves as if its resistance to acceleration increased with v.

If one identifies the work done by the external agent with the change in electromagnetic energy, $dW_{\mathrm{EM}} = \mathbf{F} \, dx$, and uses $dx = v \, dt$, one obtains

$$\mathbf{F} = \frac{1}{v} \frac{dW_{\mathrm{EM}}}{dt}.$$

Defining the inertial coefficient as the ratio of force to acceleration $a = dv/dt$ then yields the longitudinal coefficient

$$m_{\parallel} = \frac{\mathbf{F}}{a} = \frac{1}{v} \frac{dW_{\mathrm{EM}}}{dv}.$$

Abraham objected that this energy-based definition is not adapted to Kaufmann's deflection experiments, where the applied forces are predominantly transverse to the electron's velocity. A transverse force does no work, so energy considerations alone cannot fix the transverse inertial response.

In Abraham's framework one must instead start from electromagnetic momentum. For a moving electron, the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$

is nonvanishing, so that electromagnetic energy flows through the ether in and around the electron. The corresponding electromagnetic momentum density equals $\mathbf{g}_{\mathrm{EM}} = \mathbf{S}/c^2$, and the total electromagnetic momentum is

$$\mathbf{G}_{\mathrm{EM}}(v) = \frac{1}{c^2} \int \mathbf{S} \, dV = \frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{B} \, dV.$$

For uniform motion and sufficiently slow acceleration (so that no momentum is lost to radiation), the force is defined by momentum balance,

$$\mathbf{F} = \frac{d\mathbf{G}_{\mathrm{EM}}}{dt}.$$

If $\mathbf{G}_{\mathrm{EM}}(v)$ were proportional to v, the inertial response would be isotropic. For a rigid electron, however, $\mathbf{G}_{\mathrm{EM}}(v)$ is a nonlinear function of v. Writing $\mathbf{G}_{\mathrm{EM}}(v) = G(v) \, \hat{\mathbf{v}}$ and decomposing force and acceleration into components parallel and perpendicular to $\mathbf{v}$, one finds

$$\mathbf{F}_{\parallel} = \frac{dG}{dv}\,\mathbf{a}_{\parallel}, \qquad \mathbf{F}_{\perp} = \frac{G}{v}\,\mathbf{a}_{\perp}.$$

This motivates the definitions

$$m_{\parallel} = \frac{dG}{dv}, \qquad m_{\perp} = \frac{G}{v}.$$

For longitudinal forcing, the power delivered equals the rate of change of electromagnetic energy:

$$\frac{dW_{\text{EM}}}{dt} = \mathbf{F}\cdot\mathbf{v} = F_{\parallel} v.$$

Independently, momentum balance gives

$$F_{\parallel} = \frac{dG}{dt} = \frac{dG}{dv}\frac{dv}{dt}.$$

Combining the two relations yields the identity

$$\frac{dW_{\text{EM}}}{dt} = v\,\frac{dG}{dt}, \qquad \text{equivalently} \qquad dW_{\text{EM}} = v\,dG$$

for longitudinal changes of motion. Hence the longitudinal coefficient obtained from energy necessarily coincides with that obtained from momentum:

$$m_{\parallel} = \frac{1}{v}\frac{dW_{\text{EM}}}{dv} = \frac{dG}{dv}.$$

The distinction between longitudinal and transverse inertial coefficients arises because $G(v)$ is nonlinear: longitudinal forces change the *magnitude* of v and probe dG/dv, whereas transverse forces change only its *direction* and probe G/v, and these coefficients need not coincide.

G: <u>The electron is a sphere with uniform volume- or surface charge.</u>

The details of the behavior of the electron do not depend on the details of the forces that keep the electron in shape, however, they do essentially depend on such shape. If the electron is modeled as an extended, spherically symmetric charge distribution, the electric field lines are purely radial. As a consequence, in rest frame the Maxwell stresses of the electromagnetic field with isotropic configuration, their angular averages satisfy:

$$\langle E_x^2\rangle = \langle E_y^2\rangle = \langle E_z^2\rangle = \tfrac{1}{3}E^2,$$

When the electron is set into uniform motion along the x-axis, electric and magnetic stresses are mixed by the Lorentz transformation. In particular, one finds

$$\left\langle E_x'^2 + B_x'^2 \right\rangle = \gamma^2(1+\beta^2)\,\frac{1}{3}E^2$$

These stresses produce an additional momentum flow that contributes one third of the momentum density $\mathbf{g}_{\text{EM}}$ beside the non-vanishing Poynting vector. The total electromagnetic momentum $\mathbf{G}_{\text{EM}} = \int \mathbf{g}_{\text{EM}}$ can then be decomposed into a contribution arising from the transport of electromagnetic energy W_{EM} and an additional one–third contribution due to the transport of the Maxwell stresses, yielding, in the case of quasi-stationary motion with no radiation:

$$\mathbf{G}_{\mathrm{EM}} = \frac{4}{3}\frac{W_{\mathrm{EM}}}{c^2}\,\mathbf{v}.$$

Only in the case of spherical symmetry do the transverse components of the electromagnetic momentum vanish. As a result, G_y and G_z are zero, the electron experiences no torque, and the electromagnetic momentum G_x remains parallel to the direction of motion along the x–axis. If the electron were not spherical the Maxwell stresses would not be isotropic when the electron is at rest, transverse components of the momentum would be present, and the electron would experience lateral self–forces or torques even in uniform motion. This would be incompatible with Kaufmann–type experiments, in which cathode and Becquerel rays propagate rectilinearly in the absence of external forces. For $G_x \neq 0$ while $G_y = 0$ and $G_z = 0$, the electromagnetic field configuration must therefore be symmetric, that is, generated by symmetric charge distribution of the electron.

The transport of electromagnetic energy contributes to the electron's momentum density through the Poynting vector and accounts for the unit contribution, while the transport of Maxwell stresses contributes an additional one–third through momentum flux. Together, these two mechanisms account for the increase in the electromagnetic momentum density. One can then define an effective electromagnetic mass of a spherical electron:

$$\mu_0 = \frac{G_{\mathrm{EM}}}{v} = \frac{4}{3}\frac{W_{\mathrm{EM}}}{c^2}$$

In this limited sense, Abraham's definition of relativistic rigidity makes it possible to construct an apparently consistent electromagnetic model of the electron: a spherically symmetric electron with a volume charge distribution, whose inertia is attributed entirely to the resistance of its own electromagnetic field.

To calculate quantitatively how much mass grows with velocity, Abraham combines two assumptions that are individually reasonable. Abraham assumes that the electromagnetic field of a uniformly moving electron is obtained by applying the Lorentz transformation to the electrostatic field of a spherical electron at rest. Thus the electric and magnetic fields $\mathbf{E}(v)$ and $\mathbf{B}(v)$ satisfy the standard Lorentz transformation laws. After the Lorentz transformation, the electromagnetic quantities entering Abraham's energy and momentum integrals—such as E^2, $E_z^2 + c^2B_z^2$, and $\mathbf{E}\times\mathbf{B}$—are no longer isotropic. They acquire an explicit dependence on the polar angle θ through factors of the form $(1 - v^2\sin^2\theta)^{-1}$ or $(1 - v^2\sin^2\theta)^{-3/2}$.

At the same time, Abraham assumes that the electron is rigid: its spatial charge distribution is fixed and its volume is unchanged by uniform motion. In particular, the moving electron is not Lorentz-contracted along the direction of motion. The volume element dV entering the field integrals for energy and momentum is therefore the same as for the spherical electron at rest. Because the electromagnetic fields are Lorentz-transformed for a moving charge while the integration domain is not contracted, the angular dependence of the transformed fields does not cancel against any corresponding change of the electron's volume. For a spherical charge distribution this leads to an electromagnetic momentum $G(\beta)$, with $\beta = v/c$, containing logarithmic terms of the form

$$\ln\left(\frac{1+\beta}{1-\beta}\right),$$

rather than a simple factor proportional to $\gamma\beta$.

Evaluating the integrals explicitly, Abraham obtains the velocity-dependent longitudinal and transverse electromagnetic masses

$$m_\parallel \equiv \mu_s = \frac{3}{4}m_0\,\chi(\beta), \qquad \chi(\beta) = \frac{1}{\beta^2}\left\{-\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) + \frac{2}{1-\beta^2}\right\},$$

$$m_\perp \equiv \mu_r = \frac{3}{4}m_0\,\psi(\beta), \qquad \psi(\beta) = \frac{1}{\beta^2}\left\{\frac{1+\beta^2}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right\},$$

where $m_0 = \mu_0$ denotes the limiting (low-velocity) electromagnetic mass.

3 Lorentz' Deformable Electron

On the basis of hypotheses A–G, Abraham was able to calculate the electromagnetic momentum of the electron and, from it, to derive the corresponding electromagnetic inertial coefficients, namely the longitudinal and transverse masses. By 1904–1905 the resulting expressions appeared to reproduce Kaufmann's deflection experiments with satisfactory agreement, and Abraham regarded this as strong support for a purely electromagnetic account of inertia. Electron theory, however, pursued a more ambitious program: it sought to provide a unified account of the electrical and optical properties of matter. In classical electron theory, the optics of transparent bodies is treated as a special case of electrodynamics in media, corresponding to the regime in which bound electrons execute small oscillations about equilibrium positions while the body as a whole remains at rest. These forced oscillations under the influence of electromagnetic waves account for refraction and dispersion. The oscillating electron thereby furnishes a simple model of an emitting center of radiation, and the normal Zeeman effect shows that this model corresponds to physical reality for a wide class of spectral lines.

Since the velocities involved in such bound-electron oscillations are extremely small compared to the velocity of light, any velocity dependence of the mass may safely be neglected. For this reason, hypotheses E, F, and G play no essential role as long as the material body as a whole remains at rest. The situation changes in the optics of moving bodies. Stellar aberration indicates that the stationary reference system postulated in A does not participate in the Earth's orbital motion. How, then, is it possible that electrical and optical processes occurring on the Earth's surface reveal no detectable influence of this motion? This question was investigated in detail by H. A. Lorentz. He showed that the absence of observable effects of first order in $\beta = v/c$ (with $\beta \approx 10^{-4}$ for the Earth) is compatible with the fundamental hypotheses A–D of electron theory.

More delicate experiments, sensitive to effects of second order in β, yielded negative results and thereby posed serious difficulties for the theory. In order to account for these results, Lorentz formulated a system of additional hypotheses:

H: As a consequence of the Earth's motion through the ether, bodies undergo a contraction parallel to the direction of motion..

In its simplest form, this contraction may be expressed as

$$L = \frac{L_0}{\gamma}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}},$$

so that the longitudinal dimension of a moving body is reduced by the Lorentz factor relative to its rest length L_0. This hypothesis accounts for the negative result of the

Michelson interference experiment. It also explains the absence of a measurable torque on a charged condenser inclined with respect to the Earth's direction of motion, as investigated by Trouton and Noble.

I: The quasi-elastic forces that bind the electrons to their equilibrium positions undergo, as a consequence of the Earth's motion, the same modification as the electrical or molecular forces.

Hypothesis I can be rendered plausible by regarding the quasi-elastic forces themselves as electrical in nature. To explain the absence of double refraction due to the Earth's motion in bodies that are isotropic in their rest state, as established by the experiments of Lord Rayleigh and D. B. Brace, it suffices, for isotropic bodies that satisfy Maxwell's relations, to supplement hypotheses A, B, C, D, and H with hypothesis I, insofar as their optical behavior can be treated at the macroscopic, effectively non-dispersive level. In a dispersive body, however, the optical response depends on frequency because light interacts with bound electrons whose inertia now enters the description. Once the electrons' inertial response is taken into account, the effective inertial term may become direction-dependent within Abraham's rigid-electron framework. Thus, the motion of the Earth through the ether could, in principle, produce anisotropic optical effects, such as a velocity-dependent double refraction.

K: The electron filled with uniform volume- or surface charge when at rest, oblates when in motion, by contracting its diameter parallel to the direction of motion in the ratio $\sqrt{1-\beta^2} : 1$..

K replaces the rigid-shape hypotheses F and G and postulates a Lorentz contraction of an electron that is spherical at rest. To determine the electromagnetic momentum of the electron, one considers the same physical electron from the perspective of a frame moving uniformly with respect to the ether. If the electron is contracted only along the direction of motion and retains its transverse dimensions, then its spatial volume on a hypersurface $t = \text{const.}$ of that frame is

$$V = \frac{V_0}{\gamma},$$

where V_0 denotes the volume of the electron in the ether frame at a time t.

The electromagnetic momentum is obtained by integrating the electromagnetic momentum density over this volume at fixed inertial time,

$$\mathbf{G}_{\text{EM}} = \int_{V(t)} \mathbf{g}_{\text{EM}} \, dV.$$

It is convenient to distinguish the contribution to the momentum density arising from energy transport,

$$\mathbf{g}_{\text{en}} = \frac{\mathbf{S}}{c^2},$$

from the additional contribution associated with momentum flux carried by the Maxwell stresses.

Two relativistic effects enter this integration. First, Lorentz contraction contributes the factor $1/\gamma$ to the spatial volume. Second, the electromagnetic field itself is altered by the standard field transformation laws. In the instantaneous rest frame the field is purely electric, while in a frame in which the system moves with velocity $\mathbf{v}$ electric and magnetic fields are mixed. For a system moving with velocity v along the x-axis, the field components transform and electromagnetic energy density therefore becomes

$$w_{\mathrm{EM}} = \frac{1}{8\pi}\left(E^2 + B^2\right) = \frac{1}{8\pi}\left(E_{0x}^2 + \gamma^2\left(1+\beta^2\right)\left(E_{0y}^2 + E_{0z}^2\right)\right).$$

where the term proportional to β^2 originates from the magnetic field generated by the motion. As a result of the field mixing, the Poynting vector becomes nonvanishing and points along $\mathbf{v}$:

$$\mathbf{S} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}\,\gamma^2\beta\,E_{0\perp}^2\,\hat{\mathbf{v}} = \gamma^2\,\frac{\mathbf{v}}{4\pi}\,E_{0\perp}^2.$$

Hence the energy–transport contribution to the momentum density is

$$\mathbf{g}_{\mathrm{en}} = \frac{\mathbf{S}}{c^2} = \gamma^2\,\frac{\mathbf{v}}{4\pi c^2}\,E_{0\perp}^2.$$

This is, however, not the total momentum density in the extended-electron model. For a spherical electron at rest the Maxwell stress is isotropic on angular averaging; in particular. Under a boost, these stresses become anisotropic and contribute to the net momentum balance through their transport.

$$\left\langle E_x'^2 + B_x'^2\right\rangle = \gamma^2(1+\beta^2)\,\frac{1}{3}E^2$$

This stress-transport contribution amounts (after angular averaging) to an additional one-third of the energy-transport contribution, yielding locally

$$\mathbf{g}_{\mathrm{EM}} = \mathbf{g}_{\mathrm{en}} + \mathbf{g}_{\mathrm{st}} = \frac{4}{3}\,\mathbf{g}_{\mathrm{en}} = \frac{4}{3}\,\gamma^2\,\frac{w_{\mathrm{EM}}^{(0)}}{c^2}\,\mathbf{v},$$

where $w_{\mathrm{EM}}^{(0)} = E_0^2/(8\pi)$ is the rest-frame electromagnetic energy density. Integrating over the Lorentz–contracted volume then gives

$$\mathbf{G}_{\mathrm{EM}} = \int_{V_0/\gamma} \frac{4}{3}\,\gamma^2\,\frac{w_{\mathrm{EM}}^{(0)}}{c^2}\,\mathbf{v}\,dV = \gamma\int_{V_0}\frac{4}{3}\,\frac{w_{\mathrm{EM}}^{(0)}}{c^2}\,\mathbf{v}\,dV_0 = \frac{4}{3}\,\gamma\,\mu_0\,\mathbf{v},$$

with the electromagnetic rest mass defined by

$$\mu_0 = \frac{W_{\mathrm{EM},0}}{c^2}, \qquad W_{\mathrm{EM},0} = \int_{V_0} w_{\mathrm{EM}}^{(0)}\,dV_0.$$

Force is defined as the time derivative of the momentum $\mathbf{G}_{\mathrm{EM}}$. In general, the response to an applied force depends on its direction: a longitudinal force changes the magnitude of the velocity, whereas a transverse force changes only its direction. Decomposing force and acceleration accordingly yields

$$\mathbf{F}_{\parallel} = \frac{dG_{\mathrm{EM}}}{dv}\,\mathbf{a}_{\parallel}, \qquad \mathbf{F}_{\perp} = \frac{G_{\mathrm{EM}}}{v}\,\mathbf{a}_{\perp},$$

where G_{EM} is the magnitude of $\mathbf{G}_{\mathrm{EM}}$. With

$$G_{\mathrm{EM}} = \frac{4}{3}\,\gamma\,\mu_0\,v,$$

one obtains the longitudinal and transverse inertial coefficients

$$\mu_{\parallel} = \frac{dG_{\text{EM}}}{dv} = \frac{4}{3}\,\mu_0\,\frac{d(\gamma v)}{dv} = \frac{4}{3}\,\mu_0\,\gamma^3 = \frac{4}{3}\,\mu_0\frac{1}{\sqrt{(1-\beta^2)^3}},$$

$$\mu_{\perp} = \frac{G_{\text{EM}}}{v} = \frac{4}{3}\,\mu_0\,\gamma = \frac{4}{3}\,\mu_0\frac{1}{\sqrt{1-\beta^2}}$$

Thus the increase of inertia with velocity emerges (within this electromagnetic-mass framework) as a consequence of the velocity dependence of the bound field energy–momentum, together with the contraction hypothesis K.

L: <u>The masses of the electrons are of electromagnetic nature</u>.

Lorentz assumes the mass of electron is fully electromagnetic L. However, however Abraham famously questioned its compatibility with K of the deformable electron. The equivalence between a longitudinal inertial coefficient obtained from the energy balance and that obtained from the momentum balance relies on the assumption that the external force performs work solely to change the translational motion of the electron. This assumption fails when Lorentz contraction is taken as a genuine, dynamical deformation of the electron. If the electron is compressed along the direction of motion, a longitudinal force must not only increase the velocity of the centre of mass, but also do additional work against the internal stresses that maintain the electron's equilibrium shape. Part of the work supplied by the external force is therefore stored as internal (mechanical) energy associated with the deformation, rather than being converted entirely into translational kinetic energy. The work–energy balance becomes

$$\mathrm{d}W_{\text{EM,TOT}} = \mathbf{F}\cdot\mathbf{v}\ \mathrm{d}t = \mathrm{d}W_{\text{EM,TRA}} + \mathrm{d}W_{\text{EM,INT}}\ ,$$

where $W_{\text{EM,INT}}$ denotes the internal energy required to compress the electron, and $W_{\text{EM,TRA}}$ denotes the translational (electromagnetic) energy that contributes to the electromagnetic momentum. Only the translational part $W_{\text{EM,TRA}}$ contributes to the change of electromagnetic momentum. If one defines an energy-based longitudinal inertial coefficient by

$$m_{\parallel}^{(E)} = \frac{1}{v}\frac{dW_{\text{EM,TOT}}}{dv} = \frac{1}{v}\left(\frac{dW_{\text{EM,TRA}}}{dv} + \frac{dW_{\text{EM,INT}}}{dv}\right),$$

and compares it with the momentum-based longitudinal coefficient

$$m_{\parallel}^{(G)} = \frac{dG_{\text{EM}}}{dv} = \frac{1}{v}\frac{dW_{\text{EM,TRA}}}{dv},$$

then, because $\dfrac{dW_{\text{EM,INT}}}{dv} > 0$ when the electron is physically compressed, one obtains

$$m_{\parallel}^{(E)} > \ m_{\parallel}^{(G)}.$$

This mismatch reflects the physical fact that, when Lorentz contraction is interpreted dynamically (so that longitudinal forces perform internal work), the energy-based definition of longitudinal mass includes energy stored in internal stresses that the electromagnetic momentum balance does not register. Only if the electron remains undeformed (so that $\mathrm{d}W_{\text{EM,INT}} = 0$) do the energy-based and momentum-based definitions coincide.

Abraham therefore concluded that adopting hypothesis K (a deformable, Lorentz-contracted electron) forces one to postulate additional, non-electromagnetic inner forces which together with the electromagnetic forces determine the electron's shape—in contrast with the assumptions of the electromagnetic world-view. These supplementary forces must perform the work associated with contraction and they must be specified by a constitutive law; without such a law the hypothesis set A, B, C, D, E, K is incomplete. Abraham also argued that, for an oblate rotational ellipsoid of invariable form, motion parallel to the rotation axis is dynamically unstable, so there is no independent confirmation that the required supplementary forces indeed stabilize the deformable electron.

4 Poincaré Stressed Electron

Lorentz effectively associates the electron's inertial parameter with its electromagnetic rest energy, in the sense that one may write $\mu_0 = W_{\mathrm{EM},0}/c^2$ within a purely electromagnetic model. However, when the electromagnetic momentum of the moving electron is computed, an additional factor 4/3 appears. This discrepancy reveals an inconsistency that cannot be resolved within a purely electromagnetic description. Poincaré clearly identified the core difficulty: any mechanism that stabilizes the electron at rest must itself contribute to the total energy and momentum once the electron is set into motion. In Abraham's rigid–electron model, no work is performed by internal forces during acceleration. The internal electromagnetic stresses are therefore taken to be the same as in the state of rest, and mechanical equilibrium is preserved by assumption. By contrast, in the Lorentz electron the motion is accompanied by a contraction of the electron.

This deformation compresses the electromagnetic field, increasing the internal electromagnetic energy, much as in a compressed elastic medium. The Maxwell stresses therefore increase with velocity, while the constraints that hold the electron together are assumed to remain fixed. The balance that existed at rest is thus destroyed, and the electron becomes unstable. Poincaré recognized that this difficulty arises once the role of constraints is taken seriously. If the electron is in equilibrium at rest, the outward electromagnetic stresses must already be counterbalanced by mechanical stresses. For the Lorentz electron to remain stable in motion, these mechanical stresses cannot remain fixed. They must increase together with the electromagnetic stresses so as to preserve the same balance at all velocities.

M: An electron at rest is an extended, spherically symmetric charge distribution, whose mutual electrostatic repulsion is counterbalanced by non-electromagnetic mechanical stresses that hold the charge together..

By spherical symmetry, the angular averages of σ_{ij} are isotropic:

$$\left\langle E_z^2 + B_z^2 \right\rangle = \frac{1}{3}E^2.$$

Mechanical equilibrium at rest therefore requires compensating mechanical stresses that are likewise isotropic. If the electron is in equilibrium at rest, the outward electromagnetic stresses must already be counterbalanced by mechanical stresses of magnitude $-\frac{1}{3}$ of the electromagnetic pressure. Let us model them by an isotropic negative pressure (a tension) where the diagonal components of the stresses are:

$$\sigma_{xx}^{\mathrm{ME}} = -p_0, \quad \sigma_{yy}^{\mathrm{ME}} = -p_0, \quad \sigma_{zz}^{\mathrm{ME}} = -p_0 \qquad p_0^{\mathrm{ME}} = \tfrac{1}{3}\, w_{\mathrm{EM}}$$

where $w_{\rm EM}$ is the electromagnetic energy density. This is the simplest realization of Poincaré's idea: an elastic medium under uniform tension that exactly balances the outward Maxwell pressure.

If one boosts the system along the z–axis, the electromagnetic stresses no longer remain isotropic, and using the Lorentz transformations one finds:

$$\left\langle E_x'^2 + B_x'^2 \right\rangle = \left\langle E_y'^2 + B_y'^2 \right\rangle = \gamma^2 \left(1+\beta^2\right) \frac{1}{3} E^2.$$

Thus the transverse field components are enhanced by motion. As a consequence, the Maxwell stress component in the direction of motion increases relative to its rest value. Physically, the field configuration is distorted by motion, producing an anisotropic stress distribution. If the mechanical stresses were assumed to remain fixed (as in Abraham's rigid electron), equilibrium would be lost. The electromagnetic stress in the direction of motion would grow with velocity, while the compensating stresses would not.

Stability therefore requires that the stabilizing mechanical stresses cannot remain fixed. Since the moving electron is a Lorentz–contracted image of the electron at rest, the internal stresses must adjust in such a way that the contracted configuration again satisfies the condition of equilibrium. The longitudinal mechanical stress must therefore increase together with the corresponding electromagnetic stress so as to preserve balance. Only stresses that adjust consistently with the Lorentz contraction can continue to counterbalance the Lorentz–enhanced Maxwell stresses:

$$\sigma_{zz}^{\rm mech\,\prime} = -\gamma^2 \left(p_0 + \beta^2 p_0\right) = -\gamma^2 (1+\beta^2)\, p_0,$$

while the transverse components remain

$$\sigma_{xx}^{\rm mech\,\prime} = \sigma_{yy}^{\rm mech\,\prime} = -\,p_0.$$

Thus the mechanical stresses increase in the longitudinal direction by exactly the same factor as the corresponding electromagnetic stresses, but with opposite sign. The boosted system remains in equilibrium because the stress balance

$$\sigma_{zz}^{\rm EM\,\prime} + \sigma_{zz}^{\rm mech\,\prime} = 0$$

that held at rest continues to hold in motion. The stabilizing pressure adjusts so that the moving configuration remains a Lorentz-transformed equilibrium of the rest configuration.

Since a pressure p_0 does work $\delta W = -p_0\,\delta V$ when the volume changes, the Lorentz contraction necessarily produces an additional energy contribution. This energy must be included in the total energy of the moving electron. Only when this stress energy is taken into account is a consistent energy–momentum balance restored:

$$m_{\parallel}^{(E)} = \frac{1}{v}\frac{dW_{\rm TOT}}{dv} = \frac{1}{v}\left(\frac{dW_{\rm EM,TRA}}{dv} + \frac{dW_{\rm EM,INT}}{dv} - \frac{dW_{\rm ME,STR}}{dv}\right) = \frac{dG_{tot}}{dv}. \tag{2}$$

After inclusion of the Poincaré stresses, the total energy satisfies $dW_{tot} = v\,dG_{tot}$ for longitudinal motion, so that the longitudinal mass derived from energy coincides with that derived from momentum. The energy associated with these stabilizing (compressive) stresses contributes precisely the missing one–third required to restore the correct relation between energy and momentum.

5 Lagrangian Formulation of Electron Theory

Abraham proved that the equations of electromagnetism could be written in a form resembling Lagrange's equation. Assuming uniform linear motion for the electron Abraham introduces an effective Lagrangian $L_{\text{EM}}(v)$ for the single translational degree of freedom:

$$L_{\text{EM}} = T - V = \frac{1}{2}\int \frac{1}{4\pi}\left(B^2 - E^2\right) dV$$

where

$$T = \frac{1}{2}\int \frac{1}{4\pi} B^2 \, dV \qquad V = \frac{1}{2}\int \frac{1}{4\pi} E^2 \, dV.$$

According to F and E in Abraham's rigid-electron model the charge distribution and the associated electromagnetic field configuration are assumed to remain rigidly attached to the electron. External forces therefore perform work only on the translational degree of freedom, and no internal energy associated with deformation is generated. Under this assumption, the velocity-dependent electromagnetic energy $W_{\text{EM}}(v)$ and momentum $G_{\text{EM}}(v)$, computed from the field integrals, are not independent quantities, such that

$$G_{\text{EM}}(v) = \frac{dL_{\text{EM}}}{dv},$$

$$E_{\text{EM}}(v) = v\,\frac{dL_{\text{EM}}}{dv} - L_{\text{EM}}(v).$$

From these definitions it follows identically that

$$\frac{dE_{\text{EM}}}{dv} = \frac{d}{dv}\left(v\,\frac{dL_{\text{EM}}}{dv} - L_{\text{EM}}\right) = v\,\frac{d^2 L_{\text{EM}}}{dv^2} = v\,\frac{dG_{\text{EM}}}{dv}.$$

Because the rigid model excludes any additional internal energy, the explicitly computed functions $E_{\text{EM}}(v)$ and $G_{\text{EM}}(v)$ satisfy

$$\frac{dE_{\text{EM}}}{dv} = v\,\frac{dG_{\text{EM}}}{dv},$$

so that the longitudinal inertial coefficient derived from the momentum,

$$m_{\parallel}^{(G)} = \frac{dG_{\text{EM}}}{dv},$$

coincides with that derived from the energy,

$$m_{\parallel}^{(E)} = \frac{1}{v}\frac{dE_{\text{EM}}}{dv}.$$

In this sense the Lagrangian description is not an additional hypothesis but a compact reformulation of the rigid-electron dynamics already obtained from the electromagnetic field integrals. The existence of the effective function $L_{\text{EM}}(v)$ merely expresses the fact that, in the absence of deformation or internal stresses, the translational motion forms a closed mechanical system.

Lorentz, by contrast, assigns to the electromagnetic field a Lagrangian chosen so as to transform covariantly under Lorentz transformations. For the Lorentz–contracted electron according to K, the electromagnetic Lagrangian takes the form

$$L_{\text{EM}}(v) = -\frac{W^0_{\text{EM}}}{\gamma},$$

where W^0_{EM} denotes the electromagnetic energy in the rest frame of the electron. This choice ensures Lorentz covariance of the energy–momentum relations, but it does not by itself yield a dynamically stable electron, since the electromagnetic stresses are no longer in equilibrium in the state of motion.

Poincaré resolved this difficulty by introducing additional mechanical (stabilizing) stresses according to M. The total Lagrangian of the electron then becomes

$$L_{\text{TOT}}(v) = -\frac{W^0_{\text{EM}} + W^0_{\text{ME}}}{\gamma},$$

where W^0_{ME} is the rest–frame energy associated with the stabilizing stresses. Only this combined electromagnetic–mechanical Lagrangian is both Lorentz covariant and dynamically consistent. In this case, the Legendre construction yields energy and momentum that transform coherently and restores full consistency between the energy–based and momentum–based definitions of inertial mass.

The electromagnetic (canonical) momentum is obtained by differentiation,

$$G(v) = \frac{dL_{\text{TOT}}}{dv} = \frac{W_0}{c^2}\,\gamma\, v,$$

using $d(1/\gamma)/dv = -\gamma v/c^2$. The corresponding energy follows from the Legendre transform,

$$E(v) = v\,\frac{dL_{\text{TOT}}}{dv} - L_{\text{TOT}}(v) = v\,G(v) + \frac{W_0}{\gamma} = \gamma\, W_0.$$

The longitudinal inertial coefficient obtained from momentum,

$$m^{(G)}_{\parallel} = \frac{dG}{dv} = \frac{W_0}{c^2}\,\frac{d(\gamma v)}{dv} = \frac{W_0}{c^2}\,\gamma^3,$$

coincides with that obtained from energy,

$$m^{(E)}_{\parallel} = \frac{1}{v}\frac{dE}{dv} = \frac{1}{v}\frac{d(\gamma W_0)}{dv} = \frac{W_0}{c^2}\,\gamma^3.$$

Thus, once the stabilizing stresses are included in the Lagrangian, the Legendre construction guarantees both the correct transformation properties of energy and momentum and the equality of the energy–based and momentum–based definitions of longitudinal inertial mass.

6 The End of Electron Theory

In the last edition of his textbook, Abraham (1914) argues that speculative assumptions about the internal structure of the electron became unnecessary, if one, following Planck, elevates the relation

$$\mathbf{g} = \frac{\mathbf{S}}{c^2},$$

to the status of a general axiom, identifying the momentum density $\mathbf{g}$ with the energy–flux density $\mathbf{S}$ for *all* forms of energy, electromagnetic as well as mechanical. On this basis, Abraham analyzes an extended charged system (the electron) held together by internal mechanical stresses. A stress represents momentum crossing a surface per unit time, that is, a momentum flux. When such stresses are transported or give rise to internal energy circulation, they contribute to the momentum density, understood as the amount of momentum stored per unit volume. Since energy flux and momentum density are related by $\mathbf{g} = \mathbf{S}/c^2$, a system in mechanical equilibrium may sustain hidden internal energy currents with vanishing net flow. When the electron moves uniformly with velocity $\mathbf{v}$, electromagnetic energy is convected with the motion, while the mechanical stresses generate an internal flow of mechanical energy If the electron is to remain in stationary equilibrium (no net internal acceleration, no radiation), the mechanical energy flow cannot simply add to the electromagnetic one. Instead, it must compensate it.

By integrating the total energy flux through space for a spatially finite system in uniform translation, Abraham derives the global kinematical identity

$$\mathbf{G} = \frac{W}{c^2}\,\mathbf{v},$$

where W is the total energy and $\mathbf{G}$ the total momentum. This result is independent of the system's shape or constitution. It implies that the total momentum is necessarily parallel to the velocity.

N: <u>Hypothesis about the form, charge distribution and nature of mass of the electron are supererogatory</u>.

Earlier paradoxes—such as the appearance of a transverse electromagnetic momentum for nonspherical electrons—are resolved once the compensating mechanical momentum is included, since

$$\mathbf{G}^{\perp}_{\text{EM}} + \mathbf{G}^{\perp}_{\text{ME}} = 0.$$

No external torque is therefore required to maintain uniform motion. From the same relation Abraham infers the inertia of energy: the transverse mass is

$$m_r = \frac{W}{c^2},$$

and, in the rest frame,

$$m_0 = \frac{W_0}{c^2}.$$

This result holds only if all energy contributions, including those associated with stresses, are taken into account. In this way Abraham explains the earlier 4/3 discrepancy for the electromagnetic mass of a charged sphere: during motion, the mechanical energy counterflow must contribute a momentum

$$\mathbf{G}_{\text{ME}} = -\frac{1}{3}\mathbf{G}_{\text{EM}},$$

so that the total momentum satisfies

$$\mathbf{G} = \frac{W}{c^2}\mathbf{v}$$

. This is an explicit instance of what is now called hidden momentum.

Finally, Abraham reformulates the argument variationally. Assuming a Lagrangian $L(v)$ with

$$\mathbf{G} = \frac{\partial L}{\partial v}, \qquad W = v \cdot \mathbf{G} - L,$$

and imposing $\mathbf{G} = (W/c^2)\mathbf{v}$, he obtains

$$L = -m_0 c^2 \sqrt{1-\beta^2}, \qquad \beta = \frac{v}{c}.$$

From this follow the standard relativistic expressions

$$\mathbf{G} = \gamma m_0 \mathbf{v}, \qquad W = \gamma m_0 c^2,$$

and hence the familiar transverse and longitudinal masses,

$$m_{\parallel} = \gamma m_0, \qquad m_{\perp} = \gamma^3 m_0,$$

showing that the full relativistic dynamics of a particle emerges once the inertia of energy is consistently taken into account.

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